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Intrinsically extended stochastic particles with internal de Sitter symmetry

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Abstract

We give the propagation of extended particles in a generic curved spacetime. The extended particle may be a scalar system of two quarks viewed as two quantum modes. Precisely, the first mode represents the global location of the extended particle in the curved spacetime and is quantized by the geometro-stochastic method which seems to be well suited for that purpose. The other mode, which represents a relative motion and is naturally confined in a de Sitter internal spacetime, is quantized by the method of induced representations. This corresponds to the relativistic harmonic oscillator in which the interaction has been replaced by a curvature of the internal space (relativistic rotator). States of the extended particle are then defined in a Hilbert bundle structure with the direct product of the external Poincaré and internal de Sitter symmetries playing the role of the structural group. Intertwining operators are used to define propagation in one fibre as a transition amplitude between the so-called local quantum frames. Parallel transport of these frames is used to define the total propagation of the extended particle as advocated by the geometro-stochastic theory.

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1. Introduction

The currently accepted conception of hadrons is that they are composed of quarks which are deemed to be elementary particles on their own. However, the hypothesis of permanent quark confinement lacks a rigorous proof and opens the way to alternative models. One such model is the relativistic rotator based on the de Sitter dynamical group and leading to an acceptable mass–spin relation [1–3]. This model has been related to a de Sitter gauge theory describing the collective motion of an extended particle with strong interaction [4]. This theory adopted the idea of a semi-classical geometric model for extended hadrons that attributes the extension to geometric features (such as the curvature of the internal de Sitter space) rather than to the presence of constituents [5–7]. This latter model has been quantized by the geometro-stochastic method which introduces another extension related to the indivisibility of spacetime at small orders of magnitude (Planck length for instance) [8, 9].

The main purpose of the present work is to pave the way for the combination of the geometric ideas which account for confinement with all the experimental success of the quark model in a curved spacetime context. The construction is quite general and does not pretend yielding experimentally verifiable results. We shall give the mathematical structure and a general formula for the propagation of an extended scalar particle viewed as a two-quark system without specifying the unitary symmetry.

This program began with our proposal of a geometro-differential model in which the extended particle is composed of an external and an internal quantum mode [10, 11]. The external mode represents a global (mean) location of the particle in an external spacetime M with points x and the internal mode plays the role of a constituent located in an internal de Sitter spacetime V_4^R with points ξ. An analogous image has been considered in studying the Poincaré group representations of the relativistic harmonic oscillator describing a quark–antiquark system with a centre-of-mass variable x and a relative motion variable ξ [12]. In our model, the harmonic oscillator interaction is replaced by the internal curved de Sitter spacetime so that it meets the relativistic rotator with an interest in the internal local dynamics. Our first construction was based on the ideas of a quantum functional theory which adopted a conceptualization of a nonrigid extended body for the particles described by means of a physical wave u [13, 14]. This physical wave describes all the intrinsic characteristics of the particle, but does not have a probabilistic interpretation. This latter role is played by the functional wave $X[u, t]$. Compared to the conventional quantum mechanics, the functional theory replaces the point x by a wave u and the wavefunction $\psi(x, t)$ by the functional $X[u, t]$. In order to treat the abstract function u, one should adopt a realistic model. This choice is quantum in our case since the functional $X[\Psi] = \Psi(x, \xi)$ is a bilocal field describing the quantum motion of the external and the internal modes. The quantization of both modes has been carried out by an induced representation method which is based on the intertwining of the reducible configuration representation and the irreducible momentum representation with definite mass and spin [15, 16]. States of the former have been called localized and those of the latter have been interpreted as real (or material). Intertwining is then interpreted as a localization, a materialization or propagation of quantum states according to the nature of the initial and final states. We shall not rely on this interpretation in the present work but use the inducing method as a mathematical device to obtain the propagator in the internal space. We shall investigate this interpretation in forthcoming works.

In our previous works, we used a Hilbert bundle over external spacetime in which the pointlike structure of the external mode was a handicap for the curved external spacetime propagation. In the present work, we endow this latter mode with a stochastic extension in accordance with the geometro-stochastic theory to overcome the curved spacetime difficulties. In fact, the geometro-stochastic quantization provides a unified approach to nonrelativistic, relativistic and general relativistic quantum theories devoid of the inconsistencies that plague conventional theories [17, 18]. It is based on the concept of stochastic values of physical quantities rather than on sharp ones. In other words, when a measurement of a position yields a value $\mathbf{x} \in \mathbb{R}^3$ (a sharp deterministic value) it is interpreted as the real position of the particle in conventional quantum mechanics. In stochastic quantum mechanics the measurement may yield another value $\mathbf{q} \in \mathbb{R}^3$ with a confidence function (density of probability) representing the imperfectness of any apparatus. The mathematical formulation of this idea is based on the use of positive-operator-valued (POV) measures which take over the role of the projection valued

(PV) measures in conventional quantum mechanics. In the presence of a group of invariance, a POV measure leads to the notion of a system of covariance which is the generalization of the system of imprimitivity for quantum systems [19]. Accordingly, the particles are stochastically extended and described by means of square integrable functions $\psi(q, p)$ on phase space with variables (q, p). The functions $\psi(q, p)$ are defined in terms of proper state vectors $\tilde{\eta}_{q,p}$ describing test particles playing the role of microdetectors in the nonrelativistic case and used to define local quantum frames in the general relativistic case [18].

The physical justification of the present work is that the external mode experiences direct measurements and is concerned with the localization problem which seems to be solved by the geometro-stochastic theory. In contrast and because of confinement, the internal mode is observed through indirect measurements and its actual position in the internal space is not crucially relevant.

In section 2, we present a nongeometric but relativistic structure with external Poincaré and internal de Sitter symmetries. We consider the simplest scalar case and show that the propagator is a product of the external and internal pointlike propagators when there is no interaction. In section 3, we define the fibre bundle describing the states of the extended particle and give the connection representing the interaction. In section 4, we replace the geometro-stochastic and the induced representation intertwining operators by local ones acting in a single fibre. Then, the total propagation is defined as a path-integral according to the geometro-stochastic recipe which uses parallel transport of quantum frames. In section 5, we give the conclusion.

2. The external and internal spacetime symmetries

The symmetry group of the extended particle is a direct product $ISO(3,1) \otimes SO(4,1)$ of the external Poincaré group $P = ISO(3,1)$ and the internal de Sitter group $G = SO(4,1)$. The former is composed of spacetime translations $a \in T$ and Lorentz transformations $l \in \mathcal{L}$. The de Sitter group$G$ is defined as the group of transformations g which leave the quadratic form $[\xi,\xi]$ invariant in de Sitter space V_4^R(a space of constant curvature R)

$$\begin{aligned} &V_4^R = \{\xi/[\xi,\xi] = \eta_{a,b}\xi^a\xi^b = -R^2, \eta_{ab} = \mathrm{diag}(+,-,-,-,-)\} \\ &a,b = 0,1,2,3,5. \end{aligned} \tag{1}$$

To set up the construction, let us consider the Hilbert space $L^2\big(V_m^+ \times C\big)$ of square integrable functions $\Phi(k,\tilde{\zeta}^\varepsilon)$ defined on the external Poincaré momentum space (the forward mass hyperboloid) and the internal de Sitter momentum space, respectively

$$\begin{aligned} &V_m^+ = \{k/\eta_{ij}k^ik^j = m^2, k^0 > 0, \eta_{ij} = \mathrm{diag}(+,-,-,-)\} \\ &C = \left\{\tilde{\zeta}^\varepsilon/\tilde{\zeta}^\varepsilon = \begin{pmatrix}\tilde{\zeta}^i \\ -\epsilon\end{pmatrix}, \eta_{ij}\tilde{\zeta}^i\tilde{\zeta}^j = 1, \tilde{\zeta}^0 > 0, \epsilon = \pm 1\right\} \\ &i,j = 0,1,2,3. \end{aligned} \tag{2}$$

The spaces, C and V_4^R, can be viewed as homogenous spaces isomorphic to the quotient spaces of the de Sitter group

$$C \cong G/H, \qquad V_4^R \cong G/L. \tag{3}$$

The subgroup H is the little group which leaves the direction $(e_0 - e_5) = (1,0,0,0,-1)^t$ (t stands for transpose) invariant and$L$ is the internal Lorentz subgroup with elements Λ leaving the origin $\overset{0}{\xi} = \big(0,0,0,0,\frac{1}{R}\big)^t$ of V_4^R invariant. The points ξ and $\tilde{\zeta}^\varepsilon$ correspond to classes whose respective representatives are a de Sitter boost ξ_T and a Lorentz boost $\tilde{\zeta}_L I_\varepsilon$, where I_ε

is a diagonal matrix with elements $(1, \varepsilon, \varepsilon, \varepsilon, \varepsilon)$. These representatives can be related through an internal Lorentz transformation $((\tilde{\zeta}_L I_\varepsilon)^{-1}, \xi)_L$ [15, 16]

$$(\tilde{\zeta}_L I_\varepsilon)^{-1}\xi_T = ((\tilde{\zeta}_L I_\varepsilon)^{-1}\xi)_T((\tilde{\zeta}_L I_\varepsilon)^{-1}, \xi)_L. \tag{4}$$

Now we can define the direct product $\tilde{U}^\mu_m = \tilde{U}_m(a, l) \otimes \tilde{U}^\mu(\xi_T\Lambda)$ of the Poincaré and Sitter group irreducible induced momentum representations $\tilde{U}_m$ and $\tilde{U}^\mu$, by its action on the scalar bilocal field $\Phi(k, \tilde{\zeta}^\varepsilon)$. In the explicit expression of the representation $\tilde{U}^\mu_m$, we shall identify the function $\Phi(k, \tilde{\zeta}^\varepsilon)$ with the function $\Phi(k, \tilde{\zeta}_L I_\varepsilon)$, where the internal momentum variable $\tilde{\zeta}^\varepsilon$ is replaced with its representative $\tilde{\zeta}_L I_\varepsilon$ [16]

$$[\tilde{U}^\mu_m((a, l), \xi_T\Lambda)\Phi](k, \tilde{\zeta}_L I_\varepsilon) = \exp(\imath m a \cdot k)\left[\frac{[\xi, \tilde{\zeta}^\varepsilon]}{R}\right]^{\mathrm{i}\mu+\frac{3}{2}}\left\{\Phi\left(l^{-1}k, \Lambda^{-1}((\tilde{\zeta}_L I_\varepsilon)^{-1}, \xi)^{-1}_L\right)\right\}. \tag{5}$$

The parameter m stands for the mass of the external mode and $\mu \in \mathbb{R}^+$ is the parameter characterizing a unitary irreducible representation of the de Sitter group (an element of the principal series). For a fixed value of the parameter μ, the relation between the mass m' of the internal mode and the radius R is given by [16]

$$(m'R)^2 = \mu^2 + \tfrac{1}{4}; \qquad \hbar = c = 1. \tag{6}$$

The localization of the extended particle is dealt with by considering the Hilbert space $L^2\left(\Sigma \times V_4^R\right)$ of functions $\Psi(q, p, \xi)$ defined on the internal de Sitter space V_4^R and an external phase space hypersurface $\Sigma(q, p)$,

$$\Sigma(q, p) = \left\{(q, p); (q, p) \in \sigma \times V_m^+\right\}, \tag{7}$$

where σ is a space-like hypersurface in the external Minkowski spacetime M with invariant measure $d\Sigma(q, p) = p_k d\sigma^k d\Omega(p)$. The latter equals $d^3\mathbf{q}d^3\mathbf{p}$ when σ is a fixed time hyperplane ($q^0 = \text{const}$) [17]. The direct product $\hat{U} = U \otimes U^D$ of the external Poincaré phase space representation $U(a, l)$ and the internal de Sitter (induced) configuration representation $U^D(g)$ acts on the states $\Psi(q, p, \xi)$ as follows [16, 17]:

$$[\hat{U}((a, l), g)\Psi](q, p, \xi) = \Psi(l^{-1}(q - a), l^{-1}p, g^{-1}\xi). \tag{8}$$

Both representations, $U(a, l)$ and $U^D(g)$, are reducible. The irreducible component $U^{D\mu}$ of the de Sitter configuration representation is obtained by means of an intertwining operator I^μ from the momentum representation Hilbert space H^μ onto the subspace $H^{D\mu}$ of the configuration representation Hilbert space H^D. On the other hand, the irreducible component $\tilde{U}_m$ of the Poincaré phase space representation is obtained by means of a unitary intertwining operator W_η from the momentum representation Hilbert space $L^2\left(V_m^+\right)$ onto the subspace $\mathbf{P}_\eta L^2(\Sigma)$ of $L^2(\Sigma)$. The projection $\mathbf{P}_\eta$ from$L^2(\Sigma)$ onto $\mathbf{P}_\eta L^2(\Sigma)$ can be expressed by means of the so-called proper state vectors $\tilde{\eta}_{q,p}$ [17]

$$\mathbf{P}_\eta = \int_\Sigma |\tilde{\eta}_{q,p}\rangle\, \mathrm{d}\Sigma(q, p)\langle\tilde{\eta}_{q,p}|. \tag{9}$$

In the momentum representation, the proper state vectors $\tilde{\eta}_{q,p}(k)$ are obtained from functions $\tilde{\eta}(k)$ by means of a translation q and a Lorentz boost l_p,

$$\tilde{\eta}_{q,p}(k) = [\tilde{U}(q, l_p)\tilde{\eta}](k) = \exp(\imath k \cdot q)\tilde{\eta}\left(l_p^{-1}k\right). \tag{10}$$

Now we turn to the free propagation of the extended particle

$$\left[K^\mu_\eta\Psi\right](q, p, \xi) = \int \mathrm{d}\Sigma(q', p')\, \mathrm{d}\mu(\xi')K^\mu_\eta(q, p, \xi; q', p', \xi')\Psi(q', p', \xi'). \tag{11}$$

The kernel is defined as the transition probability amplitude between the improper state vectors $\eta_{q,p,\xi} = \tilde{\eta}_{q,p} \otimes \psi_\xi$, where $\psi_\xi(\xi') = \delta(\xi - \xi')$,

$$K_\eta^\mu(q, p, \xi; q', p', \xi') = \langle \eta_{q,p,\xi} | K^\mu \eta_{q',p',\xi'} \rangle \tag{12}$$

$$K^\mu = 1 \otimes \Pi^\mu. \tag{13}$$

Obviously, it is the product

$$K_\eta^\mu(q, p, \xi; q', p', \xi') = K_\eta(q, p; q', p') \Pi^\mu(\xi, \xi') \tag{14}$$

of the stochastic propagator $K_\eta(q, p; q', p') = \langle \tilde{\eta}_{q,p} | \tilde{\eta}_{q',p'} \rangle$[18] and the internal propagator $\Pi^\mu(\xi, \xi') = \langle \psi_\xi | \Pi^\mu \psi_{\xi'} \rangle$. The operator Π^μ is a composition of the intertwining operators I^μ and J^μ[16]

$$\Pi^\mu = I^\mu J^\mu, \tag{15}$$

where I^μ has been defined above and J^μ maps the reducible configuration representation Hilbert space H^D onto the irreducible momentum representation Hilbert space H^μ. The internal causal propagator $\Pi_x^{c\mu}(\xi, \xi')$ is obtained by considering the propagation of the previous mode forward in time and the propagation of the antimode backward in time

$$\Pi_x^{c\mu}(\xi, \xi') = \theta(\xi^0 - \xi'^0) \Pi^{\mu+}(\xi, \xi') + \theta(\xi'^0 - \xi^0) \Pi^{\mu-}(\xi, \xi'). \tag{16}$$

The superscripts (+) and (−) in Π^μ refer to the mode and antimode, respectively. To avoid repetitions, we shall give the explicit forms of W_η, I^μ, J^μ, K_η and $\Pi^{\mu c}$ in section 4, where we consider the propagation of the extended particle in a fibre bundle geometric structure.

3. The fibre bundles and the connection

We now proceed with the construction of a geometro-differential model for an extended particle composed of an external stochastic quantum mode with a Poincaré symmetry and an internal pointlike quantum mode with a de Sitter symmetry. Two fibre bundles are needed, namely, the momentum fibre bundle E_m^μ and the configuration fibre bundle E. Let us begin by introducing the momentum fibre bundle E_m^μ,

$$E_m^\mu\big(M, L^2\big(V_m^+ \times C\big), \tilde{U}_m^\mu\big). \tag{17}$$

The base manifold is a curved spacetime M. The typical fibre is the momentum Hilbert space $L^2\big(V_m^+ \times C\big)$ of functions $\tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon)$ defined on the external Poincaré momentum space and the internal de Sitter momentum space, respectively. The structural group $\tilde{U}_m^\mu = \tilde{U}_m(a, l) \otimes \tilde{U}^\mu(\xi_T \Lambda)$ acts on the states $\tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon)$ belonging to the fibre above a base manifold point x as defined in the previous section

$$\big[\tilde{U}_m^\mu \tilde{\Phi}_x\big](k, \tilde{\zeta}_L I_\varepsilon) = \left[\frac{[\xi, \tilde{\zeta}^\varepsilon]}{R}\right]^{\mathrm{i}\mu + \frac{3}{2}} \exp \imath m(a \cdot k) \big\{\tilde{\Phi}_x\big(l^{-1} k, \Lambda^{-1}((\tilde{\zeta}_L I_\varepsilon)^{-1}, \xi)_L^{-1}\big)\big\}. \tag{18}$$

The physical interpretation of the states $\tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon)$ is that they give the probability amplitude that the external and internal modes have the respective momenta k and $\tilde{\zeta}$ when the extended particle is localized at the mean stochastic point x of the external curved spacetime, in agreement with the stochastic theory. The inner product is given by

$$\begin{aligned} \langle \tilde{\Phi}_x | \tilde{\Phi}'_x \rangle &= \int_{V_m^+, C} \mathrm{d}\Omega(k)\, \mathrm{d}\Omega(\tilde{\zeta}) \tilde{\Phi}_x^*(k, \tilde{\zeta}_L I_\varepsilon) \tilde{\Phi}'_x(k, \tilde{\zeta}_L I_\varepsilon) \\ \mathrm{d}\Omega(k) &= \frac{\mathrm{d}^3\mathbf{k}}{2k^0}, \qquad \mathrm{d}\Omega(\tilde{\zeta}) = \delta([\tilde{\zeta}, \tilde{\zeta}])\, \mathrm{d}^4\tilde{\zeta}. \end{aligned} \tag{19}$$

The next geometric structure we need is the configuration fibre bundle

$$E\left(M, L^2\left(\Sigma \times V_4^R\right), \hat{U}\right) \tag{20}$$

whose base manifold is M and the typical fibre is the Hilbert space $L^2\left(\Sigma \times V_4^R\right)$ carrying the external Poincaré group phase space representation and the internal de Sitter group configuration representation. The direct product representation $\hat{U} = U(a,l) \otimes U^D(g)$,

$$(\hat{U}\Psi_x)(q,p,\xi) = \Psi_x(l^{-1}(q-a), l^{-1}p, g^{-1}\xi) \tag{21}$$

constitutes the structural group. The inner product is

$$\langle \Psi_x | \Psi'_x \rangle = \int_{\Sigma, V_4^R} \mathrm{d}\Sigma(q,p)\, \mathrm{d}\mu(\xi) \Psi_x^*(q,p,\xi) \Psi'_x(q,p,\xi). \tag{22}$$

The physical interpretation of the states $\Psi_x(q,p,\xi)$ is rather difficult since it involves a four-dimensional integration with respect to the internal variable ($\mathrm{d}\mu(\xi) = \delta([\xi,\xi] + R^2)\, \mathrm{d}^5\xi$). In fact, only the external part has a stochastic probabilistic interpretation for wavefunctions belonging to the irreducible component $(\mathbf{P}_\eta \otimes \mathbf{1})L^2\left(\Sigma \times V_4^R\right)$ of the external phase space representation. It corresponds to the probability of observing the fluctuations (q,p) around the mean value x [17, 18].

A classical interaction is taken into account by introducing a connection on the fibre bundle structure

$$\Gamma_T(x) = \Gamma(x) + \Gamma^R(x). \tag{23}$$

The total connection $\Gamma_T(x)$ is a sum of the external Poincaré connection $\Gamma(x)$,

$$\Gamma(x) = \mathrm{d}x^\mu \theta_{\mu k}(I^k \otimes 1) + \tfrac{1}{2}\mathrm{d}x^\mu \Gamma_{\mu kl}(x)(I^{kl} \otimes 1) \tag{24}$$

and the internal de Sitter connection $\Gamma^R(x)$,

$$\Gamma^R(x) = \tfrac{1}{2}\mathrm{d}x^\mu \Gamma^R_{\mu ab}(x)(1 \otimes I^{ab}). \tag{25}$$

The generators I^{kl} of the Poincaré group can have the following form in the phase space representation [20]:

$$\begin{aligned} I_k &= P_k = \imath\partial/\partial q^k \\ I_{kl} &= M_{kl} = (Q_k P_l - Q_l P_k) \\ Q_k &= q_k - \imath\partial/\partial p^k. \end{aligned} \tag{26}$$

The corresponding generators of de Sitter group are written in the configuration representation [8]

$$I_{ab} = \mathrm{i}\left(\xi_a \frac{\partial}{\partial \xi^b} - \xi_b \frac{\partial}{\partial \xi^a}\right). \tag{27}$$

The connection coefficient $\Gamma_{\mu kl}(x)$ plays the role of an external gravitational field while the coefficient $\Gamma^R_{\mu ab}(x)$ plays the role of an internal de Sitter gauge filed describing strong interactions between hadrons.

4. Propagation of the extended particle in a curved spacetime

The induced representation method determines the propagation of a pointlike particle by means of intertwining operators. It turns out that the generalization of these mappings to the extended particle is possible. To do so, let us consider first the mapping of states

$\tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon) \in L^2(V_m^+ \times C)$ into states $\Psi_x(q, p, \xi) \in (\mathbf{P}_\eta L^2(\Sigma) \otimes H^{D\mu}) \subset L^2(\Sigma \times V_4^R)$ realized by means of the local intertwining operator $I^\mu_{x,\eta} = W_{x,\eta} \otimes I^\mu_x$,

$$\begin{aligned} I^\mu_{x,\eta} &: L^2(V_m^+ \times C) \to (\mathbf{P}_\eta L^2(\Sigma) \otimes H^{D\mu}) \subset L^2(\Sigma \times V_4^R) \\ &\tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon) \longmapsto \left[I^\mu_{x,\eta}\tilde{\Phi}_x\right](q, p, \xi) \end{aligned} \tag{28}$$

the mappings $W_{x,\eta}$ and I^μ_x are integral transformations (see [18] and [16]) such that the operator $I^\mu_{x,\eta}$ takes the following form (I^0 is a constant):

$$\begin{aligned} \left[I^\mu_{x,\eta}\tilde{\Phi}_x\right](q, p, \xi) = {} & \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{V_m^+, C} \mathrm{d}\Omega(k)\, \mathrm{d}\Omega(\tilde{\zeta}) \exp[-\imath m(k.q)] \\ & \times \eta(k.p) \left[\frac{[\xi, \tilde{\zeta}^\varepsilon]}{R} - \mathrm{i}0\right]^{-\mathrm{i}\mu - \frac{3}{2}} I^0 \tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon). \end{aligned} \tag{29}$$

The inverse mapping is realized by means of the local operator $J^\mu_{x,\eta} = W^{-1}_{x,\eta} \otimes J^\mu_x$,

$$\begin{aligned} J^\mu_{x,\eta} &: \mathbf{P}_\eta L^2(\Sigma \times V_4^R) \to L^2(V_m^+ \times C) \\ &\Psi_x(q, p, \xi) \longmapsto [J^\mu_{x,\eta}\Psi_x](k, \tilde{\zeta}_L I_\varepsilon) \equiv \tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon) \end{aligned} \tag{30}$$

which has the following integral form (J^0 is a constant):

$$\begin{aligned} \tilde{\Phi}_x(k, \tilde{\zeta}_L I_\varepsilon) = {} & \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\Sigma, V_4^R} \mathrm{d}\Sigma(q, p)\, \mathrm{d}\mu(\xi) \exp[\imath m(k.q)] \\ & \times \eta(k.p) \left[\frac{[\xi, \tilde{\zeta}^\varepsilon]}{R} + \mathrm{i}0\right]^{\mathrm{i}\mu - \frac{3}{2}} J^0 \Psi_x(q, p, \xi). \end{aligned} \tag{31}$$

The operator $K^\mu_{x,\eta} = I^\mu_{x,\eta} J^\mu_{x,\eta} = 1 \otimes \Pi^\mu_x$,

$$\begin{aligned} K^\mu_{x,\eta} &: \mathbf{P}_\eta L^2(\Sigma \times V_4^R) \to \mathbf{P}_\eta L^2(\Sigma) \otimes H^{D\mu} \\ &\Psi_x(q', p', \xi') \longmapsto \left[K^\mu_{x,\eta}\Psi_x\right](q, p, \xi) \end{aligned} \tag{32}$$

corresponds to a propagation

$$\left[K^\mu_{x,\eta}\Psi_x\right](q, p, \xi) = \int \mathrm{d}\Sigma(q', p')\, \mathrm{d}\mu(\xi') K^\mu_{x,\eta}(q, p, \xi; q', p', \xi') \Psi_x(q', p', \xi') \tag{33}$$

in both the tangent phase space and the internal de Sitter space over the curved space point x. For the construction of the propagator in curved spacetime, we just follow the geometro-stochastic theory. In fact, the set of all the improper state vectors $\eta_{q,p,\xi}$ constitutes a local quantum frame [18] and the above propagation can be defined by

$$K^\mu_{x,\eta}(q, p, \xi; q', p', \xi') = \langle \eta_{x,q,p,\xi} | K^\mu_x \eta_{x,q',p',\xi'} \rangle. \tag{34}$$

With the connection introduced in the previous section, we define a semi-classical propagator K^μ_γ as being the local free propagator K^μ_x between the state vector $\eta_{x,q,p,\xi}$ and the parallel transported state vector $\tau(\gamma)\eta_{x',q',p',\xi'}$ along any curve γ in M,

$$K^\mu_{\gamma\eta}(x, q, p, \xi; x', q', p', \xi') = \langle \eta_{x,q,p,\xi} | K^\mu_x \tau(\gamma) \eta_{x',q',p',\xi'} \rangle. \tag{35}$$

In the scalar case, the parallel transport affects only the variables (q', p') and ξ' by path ordered elements of the Poincaré and de Sitter groups, respectively [10, 11]

$$\begin{aligned} K^\mu_{\gamma\eta}(x, q, p, \xi; x, q', p', \xi') &= K_\eta(q, p; g(\Gamma)q', g(\Gamma)p') \Pi^{\mu c}(\xi, g(\Gamma^R)\xi') \\ g(\Gamma) &= P\left(\exp\left(-\mathrm{i}\int_\gamma \Gamma(x)\right)\right) \in ISO(3, 1) \\ g(\Gamma^R) &= P\left(\exp\left(-\mathrm{i}\int_\gamma \Gamma^R(x)\right)\right) \in SO(4, 1). \end{aligned} \tag{36}$$

The symbol P means path ordering and the connections should be taken in the matrix representations of the variables they act on.

Then, we consider a foliation of the curved spacetime with space-like hypersurfaces $\sigma(t_n)$, where $x' = x_0 \in \sigma(t_0)$, $x = x_N \in \sigma(t_N)$ and $n = 0, 1, \ldots, N$. For each point x_{n-1}, we impose that x_n belongs to the intersection between $\sigma(t_n)$, the causal future of x_{n-1}, and the causal past of the endpoint $x = x_N$. We also define the average propagator $K_\eta^\mu(X_n, X_{n-1})$, with $X_n = (x_n, q_n, p_n, \xi_n)$, as the propagator $K_{\gamma\eta}^\mu(X_n, X_{n-1})$ divided by the area of that intersection. The total propagator is the limit [18]

$$K(x, q, p, \xi; x', q', p', \xi') = \lim_{\varepsilon\to 0} \int K_\eta^\mu(X_N, X_{N-1}) \times \prod_{n=N-1}^{1} \mathrm{d}\sigma(x_n)\,\mathrm{d}\Sigma(q_n, p_n)\,\mathrm{d}\mu(\xi_n) K_\eta^\mu(X_n, X_{n-1}), \tag{37}$$

where $\varepsilon = \max(t_n - t_{n-1})$. This propagator reduces to the product

$$K_{x,\eta}^\mu(q, p, \xi; q', p', \xi') = K_{x,\eta}(q, p; q', p')\Pi_x^{c\mu}(\xi, \xi') \tag{38}$$

of the external propagator

$$K_{x,\eta}(q, p; q', p') = \int_{V_m^+} \mathrm{d}\Omega(k)\tilde{\eta}^*_{q,p}(k)\tilde{\eta}_{q',p'}(k) \tag{39}$$

and the internal causal propagator $\Pi_x^{c\mu}(\xi, \xi')$,

$$\Pi_x^{c\mu}(\xi - \xi') = \theta(\xi^0 - \xi'^0)\Pi_x^{\mu+}(\xi - \xi') + \theta(\xi'^0 - \xi^0)\Pi_x^{\mu-}(\xi - \xi') \tag{40}$$

$$\Pi_x^{\mu\pm}(\xi - \xi') = \sum_\varepsilon I^{0\pm} J^{0\pm} \int_C \mathrm{d}\Omega(\tilde{\zeta}) \left\{ \left[\frac{[\xi, \tilde{\zeta}^\varepsilon]}{R} \mp \mathrm{i}0\right]^{-i\mu-\frac{3}{2}} \left[\frac{[\xi', \tilde{\zeta}^\varepsilon]}{R} \pm \mathrm{i}0\right]^{\mathrm{i}\mu-\frac{3}{2}} \right\} \tag{41}$$

when the external base manifold M is a flat Minkowski spacetime identified with all its tangent spaces (q variable) and parallel transport becomes path independent.

5. Conclusion

By adopting a geometro-stochastic description of the external mode, we have been able to construct a consistent model of scalar extended particles moving in a curved spacetime reflecting the presence of a classical gravitational field. The internal degrees of freedom are described by the internal local quantum mode moving in a de Sitter spacetime. Strong interactions between extended particles correspond to a gauge theory of the internal de Sitter symmetry while the gravitational interaction corresponds to the external Poincaré symmetry. The quantization of the internal mode has been carried out by the method of induced representations with local intertwining operators yielding an internal propagation as far as the fibre over the same external spacetime point x is concerned. The propagation between two different external spacetime points has been achieved by use of quantum frame elements $\eta_{x,q,p,\xi}$ which enable the definition of a quantum parallel transport.

In order to arrive at concrete applications of the present work, one can first test the basic idea of adopting a stochastic representation for the external mode and a pointlike representation for the internal mode in the nonrelativistic regime. We have obtained acceptable probability interpretation in that (nongeometrical) case, albeit in a general formulation [21].

One can also focus interest in the internal de Sitter space and specify the unitary symmetries, including the gauged colour symmetry acting in there. This is expected to give good results when combined with the dynamical de Sitter group. Since we are not well acquainted with unitary symmetries, we plan such applications at the latest stage of our program.

As to the propagation, one should choose a specific curved spacetime with its Poincaré connection, but much more difficult is the determination of the de Sitter connection. The latter may be deduced from a set of Einstein-like and Yang–Mills-like field equations derived from the fibre bundle structure in a nonlinear representation of the de Sitter symmetry, in addition to a wave equation for a symmetry breaking field [7, 9, 11] (see [4] for a study in the relativistic rotator context).

The present work suffers from two weak points. First, the inner product in the internal space is defined with a time integration whose probabilistic interpretation is doubtful but may be avoided by a transition to a second quantization. This kind of integration has been used in the relativistic harmonic oscillator model [12]. The second weak point may arise when considering the nonlinear representation of the de Sitter symmetry whose breaking has previously been identified with gravitation. It is clear from the present formulation that gravitation (external Poincaré symmetry) and strong interaction (internal de Sitter symmetry) have been distinguished from the outset. These two questions will be addressed in forthcoming works. Particles with spinorial internal modes (such as quarks) will be considered.

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